

SEE Transform Technique for Solving System of Linear Volterra Integro-Differential Equations of the Second Kind

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SEE Transform Technique for Solving System of Linear Volterra Integro-Differential Equations of the Second Kind

Method for finding the numerical solution of Volterra integro-differential equation. Hasan and Suleiman (2018) by applying Linear Programming problem demonstrated the numerical solution of mixed Volterra-Fredholm integral equations. Hassan et al. (2017) applied Aitken method for solving Volterra-Fredholm integral equations of the second kind with Homotopy perturbation technique. Bakodah et al. (2017) showed an efficient modification of Adomian decomposition method to solve the non-linear system of Fredholm and Volterra integro-differential equations. Chandra Guru Sekar et al. (2017) they studied the Single Term Walsh Series approach for solving the (S-LVIDE-SK) and their numerical solutions. Singh and Wazwaz (2016) they proposed a reliable technique based on Adomian decomposition method (ADM) for the numerical solution of fourth-order boundary value problems for Volterra integro-differential equations. Hassan (2011) used power functions to determine the solution of system of Fredholm integral equation of the second kind.

Where S is SEE transform operator.

If we concentrate on the SEE transform, Mansou et al. (2021) showed the usefulness of the SEE integral transform of Bessel's functions to determine the integral which includes Bessel's functions.

The main purpose of this paper is to determine the solution of the system of linear Volterra integro-differential equations of the second kind by applying SEE transform.

SEE TRANSFORM TO SOLVE SYSTEM OF LINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF THE SECOND

[illegible]

7

(6)

After simplification system (6), we get

(7)

The solution of system (7) is given as

$$S\{v_1(s)\} = \frac{\begin{vmatrix} \left\{ \begin{array}{l} S\{h_1(s)\} \\ + \frac{a_{1(m-1)}}{q^n} \\ + \frac{a_{1(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{10}}{q^{n-m+1}} \end{array} \right\} & -S\{K_{12}(s)\} \quad \dots \quad -S\{K_{1n}(s)\} \\ \left\{ \begin{array}{l} S\{h_2(s)\} \\ + \frac{a_{2(m-1)}}{q^n} \\ + \frac{a_{2(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{20}}{q^{n-m+1}} \end{array} \right\} & (q^m - S\{K_{22}(s)\}) \quad \dots \quad -S\{K_{2n}(s)\} \\ \dots & \dots \\ \left\{ \begin{array}{l} S\{h_n(s)\} \\ + \frac{a_{n(m-1)}}{q^n} \\ + \frac{a_{n(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{n0}}{q^{n-m+1}} \end{array} \right\} & -S\{K_{n2}(s)\} \quad \dots \quad (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \dots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \dots & -S\{K_{2n}(s)\} \\ \dots & \dots & \dots & \dots \\ -S\{K_{n1}(s)\} & -S\{K_{n2}(s)\} & \dots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}$$

$$S\{v_2(s)\} = \frac{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & \left\{ \begin{array}{l} S\{h_1(s)\} \\ + \frac{a_{1(m-1)}}{q^n} \\ + \frac{a_{1(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{10}}{q^{n-m+1}} \end{array} \right\} & \dots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & \left\{ \begin{array}{l} S\{h_2(s)\} \\ + \frac{a_{2(m-1)}}{q^n} \\ + \frac{a_{2(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{20}}{q^{n-m+1}} \end{array} \right\} & \dots & -S\{K_{2n}(s)\} \\ \dots & \dots & \dots & \dots \\ -S\{K_{n1}(s)\} & \left\{ \begin{array}{l} S\{h_n(s)\} \\ + \frac{a_{n(m-1)}}{q^n} \\ + \frac{a_{n(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{n0}}{q^{n-m+1}} \end{array} \right\} & \dots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \dots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \dots & -S\{K_{2n}(s)\} \\ \dots & \dots & \dots & \dots \\ -S\{K_{n1}(s)\} & -S\{K_{n2}(s)\} & \dots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}$$

$$S\{v_n(s)\} = \frac{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \dots & \left\{ \begin{array}{l} S\{h_1(s)\} \\ + \frac{a_{1(m-1)}}{q^n} \\ + \frac{a_{1(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{10}}{q^{n-m+1}} \end{array} \right\} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \dots & \left\{ \begin{array}{l} S\{h_2(s)\} \\ + \frac{a_{2(m-1)}}{q^n} \\ + \frac{a_{2(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{20}}{q^{n-m+1}} \end{array} \right\} \\ \dots & \dots & \dots & \dots \\ -S\{K_{n1}(s)\} & -S\{K_{n2}(s)\} & \dots & \left\{ \begin{array}{l} S\{h_n(s)\} \\ + \frac{a_{n(m-1)}}{q^n} \\ + \frac{a_{n(m-2)}}{q^{n-1}} \\ + \dots + \frac{a_{n0}}{q^{n-m+1}} \end{array} \right\} \end{vmatrix}}{\begin{vmatrix} (q^m - S\{K_{11}(s)\}) & -S\{K_{12}(s)\} & \dots & -S\{K_{1n}(s)\} \\ -S\{K_{21}(s)\} & (q^m - S\{K_{22}(s)\}) & \dots & -S\{K_{2n}(s)\} \\ \dots & \dots & \dots & \dots \\ -S\{K_{n1}(s)\} & -S\{K_{n2}(s)\} & \dots & (q^m - S\{K_{nn}(s)\}) \end{vmatrix}}$$

After simplification of above equations, we get the values of $S\{v_1(s)\}, S\{v_2(s)\}, \dots, S\{v_n(s)\}$. After taking the inverse ESS transform on these values, we obtain the required values of $v_1(s), v_2(s), \dots, v_n(s)$.

APPLICATIONS

In this part of the paper, some applications have been considered for illustrate the complete methodology.

Application 1. Consider the following (S-LVIDE-SK).

$$\left. \begin{aligned} v_1'(s) &= 1 + s - \frac{s^2}{2} + \frac{s^3}{3} + \int_0^s (s-t)v_1(t)dt + \int_0^s (s-t+1)v_2(t)dt \\ v_2'(s) &= -1 - 3s - \frac{3s^2}{2} - \frac{s^3}{3} + \int_0^s (s-t+1)v_1(t)dt + \int_0^s (s-t)v_2(t)dt \end{aligned} \right\} \quad (8)$$

With

$$v_1(0) = 1, v_2(0) = 1 \quad (9)$$

Operating SEE transform on system (8) and using convolution theorem of SEE transform, we have

$$\left. \begin{aligned} S\{v_1'(s)\} &= \left[-\frac{1}{2}S\{s^2\} + \frac{1}{3}S\{s^3\} \right] + \left[\frac{q^n S\{s\} S\{v_1(s)\}}{+q^n S\{s+1\} S\{v_2(s)\}} \right] \\ S\{v_2'(s)\} &= \left[-\frac{3}{2}S\{s^2\} - \frac{1}{3}S\{s^3\} \right] + \left[\frac{q^n S\{s+1\} S\{v_1(s)\}}{+q^n S\{s\} S\{v_2(s)\}} \right] \end{aligned} \right\} \quad (10)$$

Using the property “SEE transforms of derivatives” on system (10), we have

$$\left. \begin{aligned} \frac{-1}{q^n} v_1(0) + qS\{v_1(s)\} &= \left[\frac{\frac{1}{q^{n+1}} + \frac{1}{q^{n+2}}}{-\frac{1}{2} \frac{2!}{q^{n+3}} + \frac{1}{3} \frac{3!}{q^{n+4}}} \right] + \left[\frac{q^n \frac{1}{q^{n+2}} S\{v_1(s)\}}{+q^n \left(\frac{1}{q^{n+2}} + \frac{1}{q^{n+1}} \right) S\{v_2(s)\}} \right] \\ \frac{-1}{q^n} v_2(0) + qS\{v_2(s)\} &= \left[\frac{\frac{-1}{q^{n+1}} - \frac{3}{q^{n+2}}}{-\frac{3}{2} \frac{2!}{q^{n+3}} - \frac{1}{3} \frac{3!}{q^{n+4}}} \right] + \left[\frac{q^n \left(\frac{1}{q^{n+2}} + \frac{1}{q^{n+1}} \right) S\{v_1(s)\}}{+q^n \frac{1}{q^{n+2}} S\{v_2(s)\}} \right] \end{aligned} \right\} \quad (11)$$

Using equation (9) in system (11), we get

$$\left. \begin{aligned} \frac{-1}{q^n} * 1 + qS\{v_1(s)\} &= \left[\frac{\frac{1}{q^{n+1}} + \frac{1}{q^{n+2}}}{-\frac{1}{2} \frac{2!}{q^{n+3}} + \frac{1}{3} \frac{3!}{q^{n+4}}} \right] + \left[\frac{q^n \frac{1}{q^{n+2}} S\{v_1(s)\}}{+q^n \left(\frac{1}{q^{n+2}} + \frac{1}{q^{n+1}} \right) S\{v_2(s)\}} \right] \\ \frac{-1}{q^n} * 1 + qS\{v_2(s)\} &= \left[\frac{\frac{-1}{q^{n+1}} - \frac{3}{q^{n+2}}}{-\frac{3}{2} \frac{2!}{q^{n+3}} - \frac{1}{3} \frac{3!}{q^{n+4}}} \right] + \left[\frac{q^n \left(\frac{1}{q^{n+2}} + \frac{1}{q^{n+1}} \right) S\{v_1(s)\}}{+q^n \frac{1}{q^{n+2}} S\{v_2(s)\}} \right] \end{aligned} \right\} \quad (12)$$

After simplification system (12), we have

$$\left. \begin{aligned} \left(q - \frac{1}{q^2} \right) S\{v_1(s)\} - \left(\frac{1}{q^2} + \frac{1}{q} \right) S\{v_2(s)\} &= \left(\frac{1}{q^n} + \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} - \frac{1}{q^{n+3}} + \frac{2}{q^{n+4}} \right) \\ - \left(\frac{1}{q^2} + \frac{1}{q} \right) S\{v_1(s)\} + \left(q - \frac{1}{q^2} \right) S\{v_2(s)\} &= \left(\frac{1}{q^n} - \frac{1}{q^{n+1}} - \frac{3}{q^{n+2}} - \frac{3}{q^{n+3}} - \frac{2}{q^{n+4}} \right) \end{aligned} \right\} \quad (13)$$

The solution of system (13) is given by

$$\left. \begin{aligned} S\{v_1(s)\} &= \frac{\begin{vmatrix} \left(\frac{1}{q^n} + \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} - \frac{1}{q^{n+3}} + \frac{2}{q^{n+4}} \right) & - \left(\frac{1}{q^2} + \frac{1}{q} \right) \\ \left(\frac{1}{q^n} - \frac{1}{q^{n+1}} - \frac{3}{q^{n+2}} - \frac{3}{q^{n+3}} - \frac{2}{q^{n+4}} \right) & \left(q - \frac{1}{q^2} \right) \end{vmatrix}}{\begin{vmatrix} \left(q - \frac{1}{q^2} \right) & - \left(\frac{1}{q^2} + \frac{1}{q} \right) \\ - \left(\frac{1}{q^2} + \frac{1}{q} \right) & \left(q - \frac{1}{q^2} \right) \end{vmatrix}} = \frac{1}{q^n} \left[\frac{q^7 + q^6 + 2q^5 - 2q^4 - 3q^3 - 7q^2 - 4q - 4}{q^3(q^2+1)(q^3-q-2)} \right] \\ S\{v_2(s)\} &= \frac{\begin{vmatrix} \left(q - \frac{1}{q^2} \right) & \left(\frac{1}{q^n} + \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} - \frac{1}{q^{n+3}} + \frac{2}{q^{n+4}} \right) \\ - \left(\frac{1}{q^2} + \frac{1}{q} \right) & \left(\frac{1}{q^n} - \frac{1}{q^{n+1}} - \frac{3}{q^{n+2}} - \frac{3}{q^{n+3}} - \frac{2}{q^{n+4}} \right) \end{vmatrix}}{\begin{vmatrix} \left(q - \frac{1}{q^2} \right) & - \left(\frac{1}{q^2} + \frac{1}{q} \right) \\ - \left(\frac{1}{q^2} + \frac{1}{q} \right) & \left(q - \frac{1}{q^2} \right) \end{vmatrix}} = \frac{1}{q^n} \left[\frac{q^7 - q^6 - 2q^5 - 2q^4 + q^3 + 3q^2 + 4q + 4}{q^3(q^2+1)(q^3-q-2)} \right] \end{aligned} \right\} \quad (14)$$

Now

$$\begin{aligned} S\{v_1(s)\} &= \frac{1}{q^n} \left[\frac{q^7 + q^6 + 2q^5 - 2q^4 - 3q^3 - 7q^2 - 4q - 4}{q^3(q^2+1)(q^3-q-2)} \right] = \frac{1}{q^n} \left[\frac{A_1}{q} + \frac{B_1}{q^2} + \frac{C_1}{q^3} + \frac{D_1 q + E_1}{(q^2+1)} + \frac{F_1 q^2 + G_1 q + H_1}{(q^3-q-2)} \right] \\ S\{v_2(s)\} &= \frac{1}{q^n} \left[\frac{q^7 - q^6 - 2q^5 - 2q^4 + q^3 + 3q^2 + 4q + 4}{q^3(q^2+1)(q^3-q-2)} \right] = \frac{1}{q^n} \left[\frac{A_2}{q} + \frac{B_2}{q^2} + \frac{C_2}{q^3} + \frac{D_2 q + E_2}{(q^2+1)} + \frac{F_2 q^2 + G_2 q + H_2}{(q^3-q-2)} \right] \end{aligned}$$

After simple computations, we get:

$$A_1 = B_1 = 1, C_1 = 2, D_1 = E_1 = F_1 = G_1 = H_1 = 0$$

and $A_2 = 1, B_2 = -1, C_2 = -2, D_2 = E_2 = F_2 = G_2 = H_2 = 0$

Then
$$\begin{cases} S\{v_1(s)\} = \frac{1}{q^n} \left[\frac{1}{q} + \frac{1}{q^2} + \frac{2}{q^3} \right] = \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + \frac{2}{q^{n+3}} \\ \{v_2(s)\} = \frac{1}{q^n} \left[\frac{1}{q} - \frac{1}{q^2} - \frac{2}{q^3} \right] = \frac{1}{q^{n+1}} - \frac{1}{q^{n+2}} - \frac{2}{q^{n+3}} \end{cases} \quad (15)$$

Operating inverse SEE transforms on system (15), we get the required solution of system (8) with (9) as

$$\begin{aligned} v_1(s) &= S^{-1} \left\{ \frac{1}{q^{n+1}} + \frac{1}{q^{n+2}} + \frac{1}{q^{n+3}} \right\} = S^{-1} \left\{ \frac{1}{q^{n+1}} \right\} + S^{-1} \left\{ \frac{1}{q^{n+2}} \right\} + 2S^{-1} \left\{ \frac{1}{q^{n+3}} \right\} = 1 + s + s^2 \\ v_2(s) &= S^{-1} \left\{ \frac{1}{q^{n+1}} - \frac{1}{q^{n+2}} - \frac{2}{q^{n+3}} \right\} = S^{-1} \left\{ \frac{1}{q^{n+1}} \right\} - S^{-1} \left\{ \frac{1}{q^{n+2}} \right\} - 2S^{-1} \left\{ \frac{1}{q^{n+3}} \right\} = 1 - s - s^2 \end{aligned}$$

Application 2. Consider the following (S-LVIDE-SK)

$$\left. \begin{aligned} v_1''(s) &= -s^3 - s^4 + \int_0^s 3v_2(t)dt + \int_0^s 4v_3(t)dt \\ v_2''(s) &= 2 + s^2 - s^4 + \int_0^s 4v_3(t)dt - \int_0^s 2v_1(t)dt \\ v_3''(s) &= 6s - s^2 + s^3 + \int_0^s 2v_1(t)dt - \int_0^s 3v_2(t)dt \end{aligned} \right\} \quad (16)$$

$$\text{with } v_1(0) = 0, v_1'(0) = 1; v_2(0) = 0, v_2'(0) = 0; v_3(0) = 0, v_3'(0) = 0 \quad (17)$$

Operating SEE transform on system (16) and using convolution theorem of SEE transform, we have

$$\left. \begin{aligned} S\{v_1''(s)\} &= -S\{s^3\} - S\{s^4\} + q^n S\{3\}S\{v_2(s)\} + q^n S\{4\}S\{v_3(s)\} \\ S\{v_2''(s)\} &= S\{2\} + S\{s^2\} - S\{s^4\} + q^n S\{4\}S\{v_3(s)\} - q^n S\{2\}S\{v_1(s)\} \\ S\{v_3''(s)\} &= 6S\{s\} - S\{s^2\} + S\{s^3\} + q^n S\{2\}S\{v_1(s)\} - q^n S\{3\}S\{v_2(s)\} \end{aligned} \right\} \quad (18)$$

Using the property “SEE transforms of derivatives” on system (18), we have

$$\left. \begin{aligned} \frac{-v_1'(0)}{q^n} - \frac{v_1(0)}{q^{n-1}} + q^2 S\{v_1(s)\} &= \left[\begin{array}{c} -\frac{3!}{q^{n+4}} \\ -\frac{4!}{q^{n+5}} \end{array} \right] + \left[\begin{array}{c} 3q^n \frac{1}{q^{n+1}} S\{v_2(s)\} \\ +4q^n \frac{1}{q^{n+1}} S\{v_3(s)\} \end{array} \right] \\ \frac{-v_2'(0)}{q^n} - \frac{v_2(0)}{q^{n-1}} + q^2 S\{v_2(s)\} &= \left[\begin{array}{c} \frac{2}{q^{n+1}} + \frac{2!}{q^{n+3}} \\ -\frac{4!}{q^{n+5}} \end{array} \right] + \left[\begin{array}{c} 4q^n \frac{1}{q^{n+1}} S\{v_3(s)\} \\ -2q^n \frac{1}{q^{n+1}} S\{v_1(s)\} \end{array} \right] \\ \frac{-v_3'(0)}{q^n} - \frac{v_3(0)}{q^{n-1}} + q^2 S\{v_3(s)\} &= \left[\begin{array}{c} \frac{6}{q^{n+2}} - \frac{2!}{q^{n+3}} \\ +\frac{3!}{q^{n+4}} \end{array} \right] + \left[\begin{array}{c} 2q^n \frac{1}{q^{n+1}} S\{v_1(s)\} \\ -3q^n \frac{1}{q^{n+1}} S\{v_2(s)\} \end{array} \right] \end{aligned} \right\} \quad (19)$$

Using equation (17) in system (19), we get

$$\left. \begin{aligned} \frac{-1}{q^n} + q^2 S\{v_1(s)\} &= \begin{bmatrix} -\frac{3!}{q^{n+4}} \\ -\frac{4!}{q^{n+5}} \end{bmatrix} + \begin{bmatrix} 3q^n \frac{1}{q^{n+1}} S\{v_2(s)\} \\ +4q^n \frac{1}{q^{n+1}} S\{v_3(s)\} \end{bmatrix} \\ q^2 S\{v_2(s)\} &= \begin{bmatrix} \frac{2}{q^{n+1}} + \frac{2!}{q^{n+3}} \\ -\frac{4!}{q^{n+5}} \end{bmatrix} + \begin{bmatrix} 4q^n \frac{1}{q^{n+1}} S\{v_3(s)\} \\ -2q^n \frac{1}{q^{n+1}} S\{v_1(s)\} \end{bmatrix} \\ q^2 S\{v_3(s)\} &= \begin{bmatrix} \frac{6}{q^{n+2}} - \frac{2!}{q^{n+3}} \\ +\frac{3!}{q^{n+4}} \end{bmatrix} + \begin{bmatrix} 2q^n \frac{1}{q^{n+1}} S\{v_1(s)\} \\ -3q^n \frac{1}{q^{n+1}} S\{v_2(s)\} \end{bmatrix} \end{aligned} \right\} \quad (20)$$

After simplification system (20), we have

$$\left. \begin{aligned} q^2 S\{v_1(s)\} - \frac{3}{q} S\{v_2(s)\} - \frac{4}{q} S\{v_3(s)\} &= \left(\frac{1}{q^n} - \frac{3!}{q^{n+4}} - \frac{4!}{q^{n+5}} \right) \\ \frac{2}{q} S\{v_1(s)\} + q^2 S\{v_2(s)\} - \frac{4}{q} S\{v_3(s)\} &= \left(\frac{2}{q^{n+1}} + \frac{2!}{q^{n+3}} - \frac{4!}{q^{n+5}} \right) \\ -\frac{2}{q} S\{v_1(s)\} + \frac{3}{q} S\{v_2(s)\} + q^2 S\{v_3(s)\} &= \left(\frac{6}{q^{n+2}} - \frac{2!}{q^{n+3}} + \frac{3!}{q^{n+4}} \right) \end{aligned} \right\} \quad (21)$$

The solution of system (21) is given by

$$S\{v_1(s)\} = \frac{\begin{vmatrix} \left(\frac{1}{q^n} - \frac{3!}{q^{n+4}} - \frac{4!}{q^{n+5}}\right) & -\frac{3}{q} & -\frac{4}{q} \\ \left(\frac{2}{q^{n+1}} + \frac{2!}{q^{n+3}} - \frac{4!}{q^{n+5}}\right) & q^2 & -\frac{4}{q} \\ \left(\frac{6}{q^{n+2}} - \frac{2!}{q^{n+3}} + \frac{3!}{q^{n+4}}\right) & \frac{3}{q} & q^2 \end{vmatrix}}{\begin{vmatrix} q^2 & -\frac{3}{q} & -\frac{4}{q} \\ \frac{2}{q} & q^2 & -\frac{4}{q} \\ -\frac{2}{q} & \frac{3}{q} & q^2 \end{vmatrix}} = \frac{\frac{1}{q^n} [q^4 + \frac{10}{q^2} - \frac{48}{q^5}]}{q^6 + 10 - \frac{48}{q^3}} = \frac{\frac{1}{q^n} * \frac{1}{q^2} [q^6 + 10 - \frac{48}{q^3}]}{[q^6 + 10 - \frac{48}{q^3}]} = \frac{1}{q^{n+2}} \quad (22)$$

$$S\{v_2(s)\} = \frac{\begin{vmatrix} q^2 & \left(\frac{1}{q^n} - \frac{3!}{q^{n+4}} - \frac{4!}{q^{n+5}}\right) & -\frac{4}{q} \\ \frac{2}{q} & \left(\frac{2}{q^{n+1}} + \frac{2!}{q^{n+3}} - \frac{4!}{q^{n+5}}\right) & -\frac{4}{q} \\ -\frac{2}{q} & \left(\frac{6}{q^{n+2}} - \frac{2!}{q^{n+3}} + \frac{3!}{q^{n+4}}\right) & q^2 \end{vmatrix}}{\begin{vmatrix} q^2 & -\frac{3}{q} & -\frac{4}{q} \\ \frac{2}{q} & q^2 & -\frac{4}{q} \\ -\frac{2}{q} & \frac{3}{q} & q^2 \end{vmatrix}} = \frac{\frac{1}{q^n} [2q^3 + \frac{20}{q^3} - \frac{96}{q^6}]}{q^6 + 10 - \frac{48}{q^3}} = \frac{\frac{2}{q^3} [q^6 + 10 - \frac{48}{q^3}]}{q^n [q^6 + 10 - \frac{48}{q^3}]} = \frac{2!}{q^{n+3}} \quad (23)$$

$$S\{v_3(s)\} = \frac{\begin{vmatrix} q^2 & -\frac{3}{q} & \left(\frac{1}{q^n} - \frac{3!}{q^{n+4}} - \frac{4!}{q^{n+5}}\right) \\ \frac{2}{q} & q^2 & \left(\frac{2}{q^{n+1}} + \frac{2!}{q^{n+3}} - \frac{4!}{q^{n+5}}\right) \\ -\frac{2}{q} & \frac{3}{q} & \left(\frac{6}{q^{n+2}} - \frac{2!}{q^{n+3}} + \frac{3!}{q^{n+4}}\right) \end{vmatrix}}{\begin{vmatrix} q^2 & -\frac{3}{q} & -\frac{4}{q} \\ \frac{2}{q} & q^2 & -\frac{4}{q} \\ -\frac{2}{q} & \frac{3}{q} & q^2 \end{vmatrix}} = \frac{\frac{1}{q^n} [6q^2 + \frac{60}{q^4} - \frac{288}{q^7}]}{q^6 + 10 - \frac{48}{q^3}} = \frac{\frac{6}{q^4} [q^6 + 10 - \frac{48}{q^3}]}{q^n [q^6 + 10 - \frac{48}{q^3}]} = \frac{3!}{q^{n+4}} \quad (24)$$

Taking inverse SEE transforms on equations (22), (23), and (24) we obtain the required solution of the system (16) with (17) as

$$v_1(s) = S^{-1} \left\{ \frac{1}{q^{n+2}} \right\} = S$$

$$v_2(s) = S^{-1} \left\{ \frac{2!}{q^{n+3}} \right\} = s^2$$

$$v_3(s) = S^{-1} \left\{ \frac{3!}{q^{n+4}} \right\} = s^3$$

CONCLUSION

In this paper, we successfully obtained the solution of (S-LVIDE-SK) by SEE transform and the methodology completely illustrated by giving two applications. The results of these applications assert that the SEE transform is a very effective and beneficial integral transform for obtaining the exact solution of (S-LVIDE-SK). The proposed scheme can be applied to system of linear Volterra-integral equations.

APPENDICES

TABLE 1. USEFULL PROPERTIES OF SEE TRANSFORM Mansou et al. (2021),
Ajel Mansour et al. (2021)

S.N	Name of Property	Mathematical Form
1.	Linearity	$S\{cW_1(t) + dW_2(t)\} = cS\{W_1(t)\} + dS\{W_2(t)\}$
2.	Change of Scale	$S\{W(ct)\} = \frac{1}{c^{n+1}} G\left(\frac{q}{c}\right)$
3.	Shifting	$S\{e^{ct}W(t)\} = \frac{(q-c)^n}{q^n} G(q-c)$
		$S\{tW(t)\} = \left\{-\frac{n}{q} - \frac{d}{dq}\right\} G(q)$
4.	1 st Derivative	$S\{W'(t)\} = -\frac{1}{q^n} W(0) + qG(q)$
5.	2 nd Derivative	$S\{W''(t)\} = -\frac{W'(0)}{q^n} - \frac{W(0)}{q^{n-1}} + q^2G(q)$
6.	m th Derivative	$S\{W^{(m)}(t)\} = -\frac{W^{(m-1)}(0)}{q^n} - \frac{W^{(m-2)}(0)}{q^{n-1}} \dots \dots - \frac{W(0)}{q^{n-m+1}} + q^m G(q)$
7.	Convolution	$S\{W_1(t) * W_2(t)\} = q^n S\{W_1(t)\} * S\{W_2(t)\}$

TABLE 2. SEE TRANSFORM OF USEFUL FUNCTIONS
Ajel Mansour et al. (2021)

S.N	W(t)	S{W(t)} = G(q)
1.	1	$\frac{1}{q^{n+1}}$
2.	t	$\frac{1}{q^{n+2}}$
3.	t ²	$\frac{2!}{q^{n+3}}$
4.	t ^m ; m ∈ Z ⁺	$\frac{m!}{q^{n+m+1}}$
5.	e ^{ct}	$\frac{1}{q^n(q-c)}$
6.	sinct	$\frac{c}{q^n(q^2+c^2)}$
7.	cosct	$\frac{1}{q^{n-1}(q^2+c^2)}$
8.	sinhct	$\frac{c}{q^n(q^2-c^2)}$
9.	coshct	$\frac{1}{q^{n-1}(q^2-c^2)}$

TABLE 3. INVERS SEE TRANSFORM OF USEFUL FUNCTIONS
Ajel Mansour et al. (2021)

S.N	$G(q)$	$W(t) = K^{-1}\{G(q)\}$
1.	$\frac{1}{q^{n+1}}$	1
2.	$\frac{1}{q^{n+2}}$	T
3.	$\frac{2!}{q^{n+3}}$	t^2
4.	$\frac{m!}{q^{n+m+1}}; m, n \in \mathbb{Z}^+$	t^m
5.	$\frac{1}{q^n(q-c)}$	e^{ct}
6.	$\frac{c}{q^n(q^2+c^2)}$	Sinct
7.	$\frac{1}{q^{n-1}(q^2+c^2)}$	Cosct
8.	$\frac{c}{q^n(q^2-c^2)}$	Sinhct
9.	$\frac{1}{q^{n-1}(q^2-c^2)}$	Coshct

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